Improper Integrals

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Definition 1. If a function f is integrable on $[a,t]$ for any $t > a$, define the improper integral

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

if the limit converges (which means the limit exists and is finite).

If a function f is integrable on $[t,b]$ for any $t < b$, define the improper integral

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

if the limit converges.

If both $\int_a^{+\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ exist, we define

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^{+\infty} f(x)\,dx + \int_{-\infty}^{b} f(x)\,dx$$

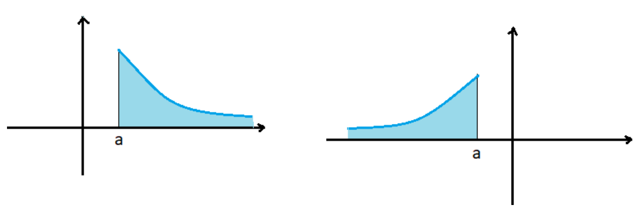

Figure 1: $\int_a^{+\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$

Remark 2. In mathematics, the opposite of converge is called diverge, and their corresponding adjective forms are convergent and divergent.

Example 3. *Discuss for which real number $p > 0$ is $\int_1^{+\infty} \frac{1}{x^p}$ convergent?*

When $p > 1$*:* $\int_1^t \frac{1}{x^p}\,dx = -\frac{1}{p-1}\frac{1}{x^{p-1}}\Big|_1^t = \frac{1}{p-1}(1-\frac{1}{t^{p-1}})$*, so*

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \frac{1}{p-1}(1-\frac{1}{t^{p-1}}) = \frac{1}{p-1}$$

When $p = 1$*:* $\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \frac{1}{x}\,dx = \ln\Big|_1^t = \ln t$*, so*

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{t\to+\infty} \ln t = +\infty$$

is divergent.

When $0 < p < 1$*:* $\int_1^t \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p}\Big|_1^t = \frac{t^{1-p}-1}{1-p}$*, so:*

$$\int_1^{+\infty} \frac{1}{x}\,dx = \int_1^{+\infty} \frac{t^{1-p}-1}{1-p} = +\infty$$

is divergent.

Example 4. *Evaluate* $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \tan^{-1} x\Big|_t^0 = 0 - \lim_{t\to-\infty} \tan^{-1} t = \frac{\pi}{2}$$
$$\int_{0}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to+\infty} \tan^{-1} x\Big|_0^t = \lim_{t\to+\infty} \tan^{-1} t - 0 = \frac{\pi}{2}$$
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{+\infty} \frac{1}{1+x^2}\,dx = \pi$$

Example 5. *Is* $\int_0^{+\infty} \sin x\,dx$ *convergent or divergent?*

$$\int_0^{+\infty} \sin x\,dx = \lim_{t\to+\infty} \int_0^t \sin x\,dx = \lim_{t\to+\infty} -\cos\Big|_0^t = 1 - \cos t$$

which is divergent.

Definition 6. If f is a function with $x = b$ a vertical asymptote, and f is integrable on $[c, t]$ for any $c < t < b$, then define

$$\int_c^b f(x)\,dx = \lim_{t \to b^-} \int_c^t f(x)\,dx$$

if the limit converges.

If f is a function with $x = a$ a vertical asymptote, and f is integrable on $[t, c]$ for any $a < t < c$, then define

$$\int_a^c f(x)\,dx = \lim_{t \to a^+} \int_t^c f(x)\,dx$$

if the limit converges.

If f is defined on $[a, c) \cup (c, b]$ with $x = c$ a vertical asymptote, $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ both exist, then define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If f is defined on (a, b), with $x = a$ and $x = b$ vertical asymptotes, $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ both exist for some $c \in (a, b)$, then define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

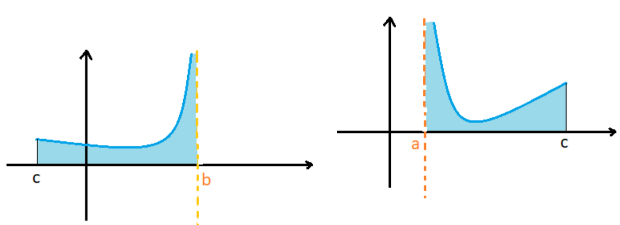

Figure 2: $\int_c^b f(x)\,dx$ and $\int_a^c f(x)\,dx$

Example 7. *Discuss for which real number $p > 0$ is $\int_0^1 \frac{1}{x^p}$ convergent?*

When $p > 1$*:* $\int_t^1 \frac{1}{x^p}\,dx = -\frac{1}{p-1}\frac{1}{x^{p-1}}\Big|_t^1 = \frac{1}{p-1}(\frac{1}{t^{p-1}} - 1)$*, so*

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \frac{1}{p-1}(\frac{1}{t^{p-1}} - 1) = +\infty$$

is divergent

When $p = 1$*:* $\int_t^1 \frac{1}{x^p}\,dx = \int_t^1 \frac{1}{x}\,dx = \ln\Big|_t^1 = -\ln t$*, so*

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} -\ln t = +\infty$$

is divergent.

When $0 < p < 1$*:* $\int_t^1 \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p}\Big|_t^1 = \frac{1-t^{1-p}}{1-p}$*, so:*

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{t\to 0^+} \frac{1-t^{1-p}}{1-p} = \frac{1}{1-p}$$

is convergent.

Example 8. *Determine if $\int_0^1 \ln x\,dx$ is convergent.*

$$\int_t^1 \ln x\,dx = x\ln x - x\Big|_t^1 = (-1) - (t\ln t - t) = t\ln t + t - 1$$

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx = \lim_{t\to 0^+} t\ln t + t - 1 = \lim_{t\to 0^+} t\ln t - 1$$

We can use the L'Hospital's Rule to determine the last limit:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-1}} = \lim_{t\to 0^+} \frac{(\ln t)'}{(t^{-1})'} = \lim_{t\to 0^+} \frac{t^{-1}}{-t^{-2}} = \lim_{t\to 0^+} (-t) = 0$$

So

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} t\ln t - 1 = 0 - 1 = -1$$

We conclude this improper integral is convergent.

Example 9. *Evaluate* $\int_0^3 \frac{1}{x-1}\,dx$ *if possible*
Let $u = x - 1$*:*

$$\int_0^3 \frac{1}{x-1}\,dx = \int_{-1}^2 \frac{1}{u}\,du = \int_{-1}^0 \frac{1}{u}\,du + \int_0^2 \frac{1}{u}\,du$$

We know $\int_0^2 \frac{1}{u}\,du$ *is divergent by Example , so we conclude* $\int_0^3 \frac{1}{x-1}\,dx$ *is divergent.*

Definition 10. f is a function defined on $(a, +\infty)$ with $x = a$ a vertical asymptote, and $\int_a^c f(x)\,dx$, $\int_c^{+\infty} f(x)\,dx$ both exist for some $c \in (a, +\infty)$, then define

$$\int_a^{+\infty} f(x)\,dx = \int_a^c f(x)\,dx + \int_c^{+\infty} f(x)\,dx$$

f is a function defined on $(-\infty, b)$ with $x = b$ a vertical asymptote, and $\int_{-\infty}^c f(x)\,dx$, $\int_c^b f(x)\,dx$ both exist for some $c \in (-\infty, b)$, then define

$$\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 11. *By Example 3 and Example 9, we conclude*

$$\int_0^{+\infty} \frac{1}{x^p}\,dx$$

is divergent for any $p > 0$*.*

Theorem 12. *(Comparison Theorem)* f *and* g *are functions with* $0 \le f(x) \le g(x)$ *on* $[a, +\infty)$*.*

(i). If $\int_a^{+\infty} g(x)\,dx$ *converges, then* $\int_a^{+\infty} f(x)\,dx$ *converges, and we have* $0 \le \int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx$

(ii). If $\int_a^{+\infty} f(x)\,dx$ *is divergent, then* $\int_a^{+\infty} g(x)\,dx$ *is divergent.*

Similar result holds also for the other types of improper integrals.

Example 13. *Determine if the improper integral* $\int_0^{+\infty} e^{-x^2}\,dx$ *is convergent.*

$$\int_0^{+\infty} e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^{+\infty} e^{-x^2}\,dx$$

and $\int_0^1 e^{-x^2}\,dx$ *exists since* e^{-x^2} *is a continuous function on* $[0,1]$, *hence integrable on* $[0,1]$.

For $\int_1^{+\infty} e^{-x^2}\,dx$, *we can use the Comparison Theorem: observe on* $[1,+\infty)$ $x \le x^2$, *so* $0 < e^{-x^2} < e^{-x}$, *and*

$$\int_1^{+\infty} e^{-x}\,dx = \lim_{t\to+\infty} \int_1^t e^{-x}\,dx = \lim_{t\to+\infty} -e^{-x}\Big|_1^t = \lim_{t\to+\infty} (e^{-1} - e^{-t}) = \frac{1}{e}$$

So $\int_1^{+\infty} e^{-x}\,dx$ *converges, which implies* $\int_1^{+\infty} e^{-x^2}\,dx$ *converges. We thus conclude* $\int_0^{+\infty} e^{-x^2}\,dx$ *converges.*